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BOTTLENECK PROBLEMS
AND DYNAMIC PROGRAMMING

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§1. Introduction.

The purpose of this paper is to indicate how the theory of dynamic programming provides a mathematical formulation and a systematic approach to an interesting and significant class of production and allocation problems~~, which we shall call "bottleneck" problems. The following two problems are typical:~~

Problem 1:

At some initial period we possess a quantity x_1 of steel and a capacity x_2 of steel production, with the privilege of dividing the quantity x_1 into three parts, y_1, y_2, y_3 where y_1 is to be used to increase the capacity, y_2 to be used together with the present capacity to produce more steel, and y_3 to remain in the stockpile. Given that this allocation and production process continues for a fixed number of time periods and given the increase in capacity determined by x_1 and y_1 and the increase in the quantity of steel determined by x_2 and y_2, the question is to determine the allocation policy which maximizes the quantity of steel in the stockpile at the end of the final period.

Problem 2:

We are engaged in the manufacture of an item which requires two component parts. We possess resources which may be used to increase the rates of production of these items. Assuming

that we know the relation between rates of production and the rate of allocation of resources, what policy of allocation of resources do we employ in order to maximize the output of complete items over a fixed time interval?

These problems in their discrete forms where decisions are made at a finite number of fixed times may in some cases be attacked by the computational techniques of the theories of linear and non-linear programming. However, not only do the simplest problems, taken over time intervals of quite reasonable length, give rise to matrices of unreasonable dimensions but, more serious from the esthetic and mathematical point of view, this approach takes no cognizance of the intrinsic structure of the process.

If we consider the problem in its continuous form (an approximation technique which almost always results in a great analytic simplification), simple assumptions of linearity yield the system of differential equations

$$\frac{dx_1}{dt} = a_1y_1, \quad x_1(0) = c_1, \tag{1}$$

$$\frac{dx_2}{dt} = -y_1-y_2 + \text{Min}\,(b_1x_1, a_2y_2) = (a_2-1)y_2-y_1, \quad x_2(0) = c_2,$$

with y_1 and y_2 subject to the restrictions

$$\begin{aligned} &(a) \quad y_1, y_2 \geq 0 \\ &(b) \quad y_1+y_2 \leq x_2 \\ &(c) \quad y_2 \leq \frac{b_2x_1}{a_1} \end{aligned} \tag{2}$$

The term $\mathrm{Min}(b_1x_1, a_2y_2)$ represents the bottleneck condition and is the mathematical equivalent of the limitation on the growth of the steel stockpile due to insufficient capacity. Our aim is now to maximize $x_2(T)$.

In this form the problem may be attacked by classical methods of the calculus of variations utilizing Lagrange multipliers. However, once again this method is too general and the particular features of the problem are overshadowed.

A third attack on problems of this genre is afforded by the very interesting techniques developed by R. Isaacs in his study of a class of continuous games. Although there are some points of contact between our methods, the conceptual bases and the analytic continuations are quite distinct. In particular, his techniques would seem to lead to grave computational difficulties.

Let us mention that a simple interpretation of the second problem above leads to the mathematical question of maximizing $\int_0^T \mathrm{Min}(x,y)dt$ over all f_1 and f_2 satisfying $0 \leq f_1, f_2 \leq M$, $\int_0^T (f_1+f_2)dt \leq c_3$, where

$$\begin{aligned} \frac{dx}{dt} &= a_1x + a_2y + f_1, & x(0) &= c_1 \\ \frac{dy}{dt} &= a_3x + a_4y + f_2, & (0) &= c_2. \end{aligned} \tag{3}$$

§2. The Dynamic Programming Approach.

We shall consider only the first problem in discussing the application of dynamic programming techniques. For a brief discussion of the theory and some typical problems we refer to [1], [2].

Our first step is to imbed the problem within a continuous family by taking the quantities x_1, x_2 and T, the duration of the process, as the basic state variables. The effect of any decision, where by a decision we mean a choice of y_1, y_2 and y_3, will be to transform these state variables into a similar triple. The criterion function Max $x_2(T)$ may be written $f(x_1,x_2,T)$. Using the intuitive and obvious property that an optimal policy has the property that its continuation after any initial decisions must also be optimal with respect to the new state variables, we obtain the characteristic functional equation of dynamic programming

$$\begin{aligned} f(c_1,c_2,s+t) &= \operatorname*{Max}_{D[0,s]} f(x_1(s),x_2(s),t), \\ f(c_1,c_2,0) &= c_2, \quad c_1,c_2 \geq 0, \end{aligned} \tag{1}$$

where x_1 and x_2 satisfy (1) of o1 and by $\operatorname*{Max}_{D[0,s]}$ we mean that the maximum is to be sought over all y_1 and y_2 which satisfy the restrictions in (2) of §2 over $[0,s]$. It is quite easy to prove by elementary arguments that the maximum is assumed. Furthermore, it follows readily from (1) that the solution is unique. For setting $t = 0$, we obtain, using the initial condition

$$f(c_1,c_2,s) = \operatorname*{Max}_{D[0,s]} f(x_1(s),x_2(s),0) = \operatorname*{Max}_{D[0,s]} x_2(s) \tag{2}$$

Hence the functional equation is equivalent to the original process. Alternatively, it is easy to establish directly by successive

approximations that (1) has a solution.

Assuming that f has well-behaved partial derivatives with respect to the basic variables, we obtain via a limiting process, as $s \longrightarrow 0$, the partial differential equation

$$\frac{\partial f}{\partial t} = \max_{D[0]} \left[a_1 y_1 \frac{\partial f}{\partial c_1} + ((a_2-1)y_2-y_1)\frac{\partial f}{\partial c_2} \right] \tag{3}$$

where by $D[0]$ we mean the region in y_1, y_2 space described by

$$0 \leq y_1, y_2, \qquad y_1+y_2 \leq c_2, \qquad y_2 \leq b_1 c_1/a_2 \tag{4}$$

Since it seems rather difficult to prove directly that the solution of (1) possesses the requisite continuity properties, an indirect approach is used. It is first shown without difficulty that any solution of (3) satisfies (2). Having obtained a solution of (1) by the use of (3), the uniqueness theorem tells us that it is the solution of (1).

Turning to the task of finding solutions of (3) we first consider the case of small t where the solution is immediate. Having obtained f for small t, the partial differential equation is now utilized to deduce the form of solution as t grows. Since this is a "bootstrap" method it is necessary to verify with great care that the f obtained in this fashion is actually a solution of (3). Having found f we can now determine whether or not the optimal policy is unique. It turns out to be so in this case.

§4. An Outline of the Analytic Procedure.

Let us give a short sketch of the steps to be followed in obtaining the solution of the maximization problem. We shall consider the most complicated case where no capacity restriction exists initially, i.e., $c_2 < b_1c_1/a_2$. For processes of short duration the solution is clear, $y_1 = 0$, $y_2 = x_2$, yielding $f = c_2e^{(a_2-1)t}$. This policy is pursued until a bottleneck develops. Using the above allocation program, this will occur as soon as t exceeds $T_1 = (\log b_1c_1/a_2c_2)/(a_2-1)$.

In order to obtain the solution for times greater than T_1, we rewrite the right side of (3) of §3 in the form

$$\frac{\partial f}{\partial t} = \operatorname*{Max}_{D[0]} \left[y_1\left(a_1 \frac{\partial f}{\partial c_1} - \frac{\partial f}{\partial c_2}\right) + (a_2-1) \frac{\partial f}{\partial c_2} \; y_2 \right]. \qquad (1)$$

The location of the maximizing point $(y_1(0), y_2(0))$ will depend upon the sign and magnitude of the coefficients of y_1 and y_2. For $t < T_1$, the coefficient of y_1 is uniformly negative, while that of y_2 is uniformly positive.

Proceeding on the assumption of continuity, we suspect that the solution for t slightly greater than T_1 will have the form $y_1 = 0$, $y_2 = x_2$ for $0 \le s \le T_1$ and $y_1 = 0$, $y_2 = b_1x_1/a_2$ (due to the capacity restriction) for $T_1 \le s \le t$.

The function f will now have the form

$$f = \frac{b_1c_1}{a_2} + (t-T_1)\,\frac{(a_2-1)}{a_2}\,b_1c_1 \qquad (2)$$

In order to find out how long this policy endures, we apply (3) of §3 at the point $s = T_1$. Starting from this point we see that f has the form $c_2' + (a_2-1)\, b_1 c_1'(t-T_1)/a_2$.

The coefficient of y_1 in (1) is $a_1\, \partial f/\partial c_1' - \partial f/\partial c_2'$ $= a_1 b_1 (a_2-1)(t-T_1)/a_2 - 1$. This is 0 at $t-T_1 = a_2/a_1 b_1(a_2-1) = T^*$. This shows that this modified policy is optimal for $T_1 \leq t \leq T_1 + a_2/a_1 b_1(a_2-1)$.

For larger t, there will be an interior interval during which $y_1 = x_2 - b_1 x_1/a_2$. This corresponds to allocating steel to increase the capacity of steel mills. For large t, the optimal policy has the form

$$\begin{aligned} &y_1 = 0, \quad y_2 = x_2, \quad 0 \leq s \leq T_1 \\ &y_1 = x_2 - \frac{b_1 x_1}{a_2}, \quad y_2 = \frac{b_1 x_1}{a_2}, \quad T_1 \leq s \leq t - T \qquad (3) \\ &y_1 = 0, \quad y_2 = \frac{b_1 x_1}{a_2}, \quad t - T^* \leq s \leq t, \end{aligned}$$

where T^* is as above.

Since we have pursued a bootstrap method it is essential as mentioned above that this solution be verified. With application to inter-industry problems in mind, it is vital that this verification be analytic rather than numerical. The details of the verification are not trivial and we shall postpone any discussion until a future time.

§5. Discussion.

The method we have presented in skeleton form is applicable to a large class of problems related to the maximization of functions

or functionals depending upon the solutions of differential equations. It is also applicable to various classes of multi-stage continuous games, such as games of survival.

If—as is necessary in more realistic mathematical models dealing with the production of capital goods—time lags are taken into account, the complexity of the problem increases.

A complete and detailed treatment of the above problem will be presented subsequently, together with a discussion of extensions in the directions just cited.

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